

Exploring nonlinear wave mixing with a mass-spring simulator

Explorando mistura não linear de ondas com um simulador massa-mola

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We investigate nonlinear wave-mixing processes using a classical nonlinear mass-spring oscillator driven by one or two external periodic forces. By introducing quadratic and cubic nonlinear terms in the restoring force, we establish an analogy between the frequency-mixing structure of the oscillator dynamics and second- and third-order nonlinear optical responses, rather than a complete reproduction of the optical dynamics. Using Fourier-domain arguments, we show how the nonlinear terms couple different spectral components and generate new frequencies through processes analogous to second-harmonic generation, optical rectification, sum- and difference-frequency generation, third-harmonic generation, and four-wave mixing. Numerical simulations were performed with an interactive simulator from the SimuFísica platform using a fourth-order Runge-Kutta algorithm. The results reveal the emergence of harmonic generation, beat frequencies, DC components, four-wave-mixing sidebands, cascaded spectral generation, and frequency-comb-like structures.

Keywords: Nonlinear optics, Wave mixing, Mass-spring oscillator, Nonlinear oscillator, Computer simulation, Physics education.

Investigamos processos não lineares de mistura de ondas utilizando um oscilador massa-mola não linear clássico excitado por uma ou duas forças periódicas externas. Ao introduzir termos não lineares quadrático e cúbico na força restauradora, estabelecemos uma analogia entre a estrutura de mistura de frequências da dinâmica do oscilador e as respostas ópticas não lineares de segunda e terceira ordem, em vez de uma reprodução completa da dinâmica óptica. Utilizando argumentos no domínio da frequência, mostramos como os termos não lineares acoplam diferentes componentes espectrais e geram novas frequências por meio de processos análogos à geração de segundo harmônico, retificação óptica, geração de soma e diferença de frequências, geração de terceiro harmônico e mistura de quatro ondas. Simulações numéricas foram realizadas com um simulador interativo da plataforma SimuFísica utilizando o algoritmo de Runge-Kutta de quarta ordem. Os resultados revelam o surgimento de geração de harmônicos, batimentos, componentes DC, bandas laterais de mistura de quatro ondas, geração espectral em cascata e estruturas semelhantes a pentes de frequência.

Palavras-chave: Óptica não linear, Mistura de ondas, Oscilador massa-mola, Oscilador não linear, Simulação computacional, Ensino de física.

1. Introduction

Nonlinear optics studies phenomena associated with the propagation of intense electromagnetic radiation in material media, where the optical response of the medium ceases to be proportional to the incident electric field. Since the development of the first lasers in the 1960s, sufficiently intense electromagnetic fields have enabled the observation of several nonlinear processes, including harmonic generation, optical rectification [1], and four-wave mixing (FWM). These phenomena have applications in several areas of physics and engineering, including nonlinear spectroscopy, wavelength conversion in optical telecommunications [2], precision metrology, and optical frequency-comb generation [3, 4].

The processes underlying these applications, harmonic generation, sum- and difference-frequency generation, and four-wave mixing, are precisely those explored, in mechanical analogy, throughout the present work.

One of the best-known classical descriptions of the interaction between radiation and matter is the Lorentz model, in which electrons bound to atoms are treated as damped harmonic oscillators driven by an oscillating electric field [5]. In this model, the electron dynamics exhibits a strong mathematical analogy with that of a classical mass-spring system subjected to a periodic external force. The introduction of nonlinear terms in the restoring force allows one to qualitatively describe several phenomena typical of nonlinear optics, establishing a conceptual connection between classical mechanical systems and optical wave-mixing processes.

This analogy has been explored in several works devoted to the teaching or description of nonlinear

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optical phenomena using classical mass-spring systems [6, 7]. However, the physical mechanism and scope differ substantially from ours. Ref. [6] considers a two-mode parametric spring-pendulum system, whose optical analogy is restricted to three-wave parametric interactions ($\omega_3 = \omega_1 \pm \omega_2$). Here, the nonlinearity instead arises from a polynomial expansion of the restoring force of a single-mode oscillator, analogous to the nonlinear polarization expansion [Eq. (9)], allowing a single unified model to reproduce a broader family of second- and third-order processes (second- and third-harmonic generation, optical rectification, sum- and difference-frequency generation, FWM, and cascaded comb-like structures) not addressed in Ref. [6]. The simulator belongs to the same SimuFísica platform as our earlier works [8–10], none of which addresses this wave-mixing analogy.

In this work, we investigate the dynamics of a nonlinear mass-spring oscillator subjected to the simultaneous action of two sinusoidal external forces, emphasizing its connection with wave-mixing processes in nonlinear optics. We use an interactive computational simulator from the SimuFísica platform to numerically investigate harmonic generation, wave mixing, and frequency-comb-like spectral structures.

The correspondence considered here is restricted to the generation and mixing of spectral components produced by the nonlinear response. Other important aspects of four-wave mixing in real optical systems, such as propagation through the nonlinear medium, wave-vector conservation, and phase-matching conditions, are beyond the scope of the present study. We stress that, in real nonlinear-optical media, efficient frequency generation additionally requires the phase-matching condition to be satisfied and the generated frequencies to lie within a transparency window of the material. The mechanical model considered here has no counterpart to these requirements, since it involves no spatial propagation.

Beyond its connection with contemporary nonlinear-optics phenomena, the approach presented here has pedagogical potential for advanced undergraduate and introductory graduate courses in electromagnetism, optics, nonlinear dynamics, and computational modeling. Concepts such as harmonic generation, four-wave mixing, and optical frequency combs are often introduced through abstract expansions of nonlinear susceptibilities and frequency-domain formalisms. The analogy with a mass-spring oscillator allows these processes to be visualized in a familiar mechanical system, facilitating the connection between the temporal evolution of the oscillator and its frequency-domain response.

This paper is organized as follows. In Sec. 2, we present the theoretical model of the nonlinear oscillator and discuss the origin of the spectral components generated by the quadratic and cubic restoring-force terms. In Sec. 3, we present numerical examples obtained with the simulator, including processes analogous to second-harmonic

generation and four-wave mixing. Finally, in Sec. 4, we present the conclusions of this work.

2. Nonlinear Oscillator Model

We begin with the classical Lorentz model for electrons bound to atoms [5], in which the electron-nucleus interaction is approximated by a harmonic restoring force:

$$F_{\text{binding}} = -kx = -m\omega_0^2 x, \quad (1)$$

where m is the electron mass, x represents its displacement relative to the equilibrium position, k is the effective restoring-force constant, and ω_0 is the natural oscillation frequency of the system. To account for dissipative mechanisms, a phenomenological damping term is also introduced:

$$F_{\text{damping}} = -\gamma \frac{dx}{dt}, \quad (2)$$

where γ is the damping coefficient. The incident electromagnetic wave is assumed to have the form

$$E(t) = E_0 \cos(\omega_1 t), \quad (3)$$

with amplitude E_0 and angular frequency ω_1 .

The collective response of the medium to the incident radiation is described by the electric polarization P , defined as the induced electric dipole moment per unit volume:

$$P(t) = -Nex(t), \quad (4)$$

where e is the magnitude of the electron charge and N is the atomic density. To relate $P(t)$ explicitly to the driving field, we return to the electron equation of motion. Newton's second law, combining the elastic and damping forces of Eqs. (1) and (2) with the driving force $-eE(t)$ exerted by the field on the electron (charge $-e$), gives

$$m\ddot{x} + \gamma\dot{x} + kx = -eE(t). \quad (5)$$

Seeking a steady-state solution oscillating at the driving frequency, we write $E(t) = \text{Re}[E_0 e^{-i\omega_1 t}]$ and $x(t) = \text{Re}[x_0 e^{-i\omega_1 t}]$. Since Eq. (5) is linear, it suffices to work directly with the complex amplitudes, recovering the physical (real-valued) quantities by taking the real part only at the end. Substituting into Eq. (5) and canceling the common factor $e^{-i\omega_1 t}$ gives

$$x_0 = \frac{-eE_0}{k - m\omega_1^2 - i\gamma\omega_1}. \quad (6)$$

Substituting x_0 into Eq. (4), the polarization becomes directly proportional to the driving field, $P(t) = [Ne^2/(k - m\omega_1^2 - i\gamma\omega_1)] E(t)$. That is, one obtains the well-known linear response

$$P(t) = \epsilon_0 \chi E(t), \quad (7)$$

where ϵ_0 is the vacuum permittivity and χ is the electric susceptibility of the medium, identified from the derivation above as

$$\chi(\omega_1) = \frac{Ne^2/(m\epsilon_0)}{\omega_0^2 - \omega_1^2 - i\gamma\omega_1/m}, \quad (8)$$

the standard Lorentz-oscillator susceptibility [5], where ω_1 is the frequency of the driving field introduced in Eq. (3). In general, $\chi(\omega_1)$ is complex and frequency-dependent, its real and imaginary parts being associated, respectively, with the refractive index and with the optical absorption of the medium.

Equation (7) shows that, in the linear regime, the polarization oscillates at the same frequency as the applied electric field. Microscopically, this means that the induced atomic dipoles execute harmonic oscillations driven by the incident electromagnetic wave. Since accelerated charges emit electromagnetic radiation, the oscillating electrons re-radiate electromagnetic fields at the same frequency as the external excitation. The coherent superposition of the radiation emitted by all dipoles in the material gives rise to the electromagnetic wave emitted by the medium, characterizing the linear optical response of the system.

The same mechanical structure derived above, that is, a linear driven oscillation mapped onto the polarization through Eq. (4), is generalized, in what follows, to a nonlinear restoring force. The development of high-intensity coherent sources made experimentally accessible regimes in which the optical response of materials becomes nonlinear. Under these conditions, nonlinear contributions to the polarization can no longer be neglected, giving rise to characteristic nonlinear-optical phenomena such as harmonic generation, wave mixing, and optical rectification.

Processes such as second-harmonic generation, in which the medium emits radiation at twice the incident frequency, can only be described when the electric polarization is written as a power-series expansion of the electric field [3]:

$$P(t) = \epsilon_0 \left[\chi^{(1)} E(t) + \chi^{(2)} E^2(t) + \chi^{(3)} E^3(t) + \dots \right], \quad (9)$$

where $\chi^{(1)}$ corresponds to the linear susceptibility introduced in Eq. (7), while $\chi^{(2)}$ and $\chi^{(3)}$ are the second- and third-order nonlinear susceptibilities, respectively. In general, higher-order susceptibilities are much smaller than the linear contribution, so their effects become significant only in regimes of sufficiently intense electric fields. For simplicity, and as is common in introductory treatments [3], the susceptibilities $\chi^{(n)}$ in Eq. (9) are treated here as scalars. In general, they are tensors relating the vector components of P and E , a distinction that is not essential for the pedagogical purposes of the present work.

2.1. The nonlinear model

Our model consists of a nonlinear mass-spring oscillator subjected to the simultaneous action of two sinusoidal external forces. In analogy with the nonlinear expansion of the electric polarization introduced in Eq. (9), we generalize the restoring force of the spring by including quadratic and cubic nonlinear terms:

$$F_{el} = -kx - k_2x^2 - k_3x^3, \quad (10)$$

where k is the linear spring constant, while k_2 and k_3 characterize the nonlinear contributions to the restoring force. The quadratic term breaks the inversion symmetry of the elastic force, analogously to what occurs in nonlinear optical media with nonvanishing second-order susceptibility $\chi^{(2)}$ [3]. Within the electric-dipole approximation, $\chi^{(2)} \neq 0$ requires a non-centrosymmetric medium. The mechanical model considered here reproduces only this symmetry restriction on the allowed nonlinear response. It is not intended to reproduce the microscopic (e.g., crystalline) origin of $\chi^{(2)} \neq 0$ in real optical materials. We also note that, when $k_3 = 0$, the corresponding potential energy, $U(x) = \frac{1}{2}kx^2 + \frac{1}{3}k_2x^3$, is not globally bounded from below, so that the purely quadratic model is only locally valid around the equilibrium position.

The external forces are assumed to have the form

$$F_i(t) = F_i \cos(\omega_i t), \quad (11)$$

where F_i and ω_i represent the amplitude and angular frequency of the i th external excitation, respectively. A single monochromatic force ($F_2 = 0$) is sufficient to generate harmonics and a DC (direct-current, i.e., zero-frequency) component through the quadratic and cubic terms, since these processes involve only ω_1 . However, cross-mixing processes such as sum- and difference-frequency generation and four-wave mixing require spectral components at two distinct frequencies to combine. This is the reason for considering the simultaneous action of two external forces, $F_1(t)$ and $F_2(t)$, at frequencies ω_1 and ω_2 , analogous to two monochromatic beams incident on a nonlinear optical medium. Figure 1 illustrates the analogy between the mechanical system considered here and a nonlinear optical medium driven by two monochromatic electromagnetic fields.

Starting from Newton's second law and including a phenomenological damping term proportional to the velocity, we obtain the differential equation governing the system dynamics:

$$m\ddot{x} + \gamma\dot{x} + kx + k_2x^2 + k_3x^3 = F_1(t) + F_2(t). \quad (12)$$

The presence of nonlinear terms causes different spectral components of the motion to become coupled, allowing new frequencies to be generated from the frequencies of the external forces. In the following subsections, we separately analyze the cases dominated by the quadratic

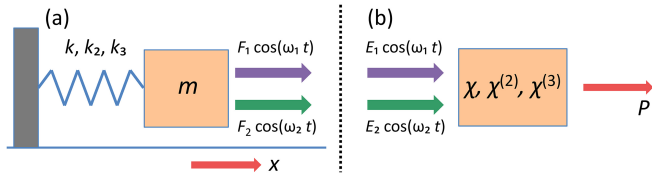


Figure 1: (a) Illustration of the model: a classical anharmonic oscillator subjected to bichromatic excitation. (b) Optical analog: two lasers incident on a nonlinear medium, generating new radiation associated with the polarization P .

and cubic nonlinearities, emphasizing their connection with wave-mixing processes in nonlinear optics. We stress that the present model is zero-dimensional, describing only the temporal generation and mixing of frequency components, with no spatial propagation (see Introduction).

2.2. Frequency mixing induced by the quadratic term

Let us first consider the case in which only the quadratic nonlinearity is present:

$$m\ddot{x} + \gamma\dot{x} + kx + k_2x^2 = F_1 \cos(\omega_1 t) + F_2 \cos(\omega_2 t). \quad (13)$$

Although approximate perturbative solutions can be obtained for this equation [3], exact analytical solutions are generally not available. To understand the origin of the frequencies generated by the nonlinearity, it is particularly convenient to analyze the system in the frequency domain.

We write the mass position in terms of its inverse Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}(\omega) e^{-i\omega t} d\omega, \quad (14)$$

where $\tilde{x}(\omega)$ represents the Fourier transform of $x(t)$. The time derivatives become

$$\dot{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\omega) \tilde{x}(\omega) e^{-i\omega t} d\omega, \quad (15)$$

$$\ddot{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-\omega^2) \tilde{x}(\omega) e^{-i\omega t} d\omega. \quad (16)$$

Using the exponential representation of the driving terms,

$$\cos(\omega_j t) = \frac{1}{2} (e^{i\omega_j t} + e^{-i\omega_j t}), \quad (17)$$

and substituting the previous expressions into Eq. (13), we obtain, in the frequency domain,

$$D(\omega) \tilde{x}(\omega) + \frac{k_2}{m} \tilde{x}^2(\omega) = \frac{\pi F_1}{m} [\delta(\omega - \omega_1) + \delta(\omega + \omega_1)] + \frac{\pi F_2}{m} [\delta(\omega - \omega_2) + \delta(\omega + \omega_2)], \quad (18)$$

where we define

$$D(\omega) = -\omega^2 - i\frac{\gamma}{m}\omega + \frac{k}{m}. \quad (19)$$

The nonlinear term involves the Fourier transform of $x^2(t)$, given by the convolution [11]

$$\tilde{x}^2(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}(\omega') \tilde{x}(\omega - \omega') d\omega', \quad (20)$$

that is, $\tilde{x}^2 = \tilde{x} \otimes \tilde{x}$. While the linear term $D(\omega) \tilde{x}(\omega)$ acts independently on each spectral component, the convolution introduces coupling between different frequencies of the system.

Within the linear approximation, and considering only the forced steady-state regime, ignoring for now the freely decaying transient response discussed in Sec. 3, the oscillator response is expected to contain only the frequencies of the external forces:

$$\tilde{x}(\omega) \sim a_1 \delta(\omega - \omega_1) + a_1^* \delta(\omega + \omega_1) + a_2 \delta(\omega - \omega_2) + a_2^* \delta(\omega + \omega_2). \quad (21)$$

Substituting these spectral components into the convolution of Eq. (20), we observe that new frequencies are generated through the sum and difference of the original frequencies. For example, the combination of the ω_1 component $[a_1 \delta(\omega - \omega_1)]$ with itself produces

$$\tilde{x}^2(\omega) \propto \int_{-\infty}^{\infty} \delta(\omega' - \omega_1) \delta(\omega - \omega' - \omega_1) d\omega', \quad (22)$$

which imposes $\omega = 2\omega_1$. A spectral component therefore emerges at twice the fundamental frequency, corresponding to the second harmonic. It is worth noting that while representing the monochromatic response in terms of Dirac delta functions is mathematically convenient, it represents an idealization of a steady-state regime.

Similarly, cross terms involving ω_1 and ω_2 produce components at

$$\omega = \omega_1 + \omega_2 \quad (23)$$

and

$$\omega = |\omega_1 - \omega_2|, \quad (24)$$

corresponding, respectively, to sum- and difference-frequency generation processes.

Since the generated frequencies may themselves participate again in the convolution process, the resulting spectrum may, in principle, contain integer linear combinations of the fundamental frequencies:

$$\omega = n\omega_1 + m\omega_2, \quad n, m = 0, \pm 1, \pm 2, \dots \quad (25)$$

This does not imply that every such combination necessarily appears with observable amplitude: higher-order combinations (larger $|n| + |m|$) generally arise from higher-order cascaded mixing processes and are progressively weaker.

2.3. Frequency mixing induced by the cubic term

Let us now consider the case in which only the cubic nonlinearity is present:

$$m\ddot{x} + \gamma\dot{x} + kx + k_3x^3 = F_1 \cos(\omega_1 t) + F_2 \cos(\omega_2 t), \quad (26)$$

corresponding to a Duffing-type equation [12, 13] subjected to bichromatic excitation. In the frequency domain, the equation becomes

$$D(\omega)\tilde{x}(\omega) + \frac{k_3}{m}\tilde{x}^3(\omega) = \frac{\pi F_1}{m} [\delta(\omega - \omega_1) + \delta(\omega + \omega_1)] + \frac{\pi F_2}{m} [\delta(\omega - \omega_2) + \delta(\omega + \omega_2)], \quad (27)$$

where

$$\tilde{x}^3(\omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{x}(\omega') \tilde{x}(\omega'') \tilde{x}(\omega - \omega' - \omega'') d\omega' d\omega'', \quad (28)$$

represents a triple convolution, $\tilde{x}^3 = \tilde{x} \otimes \tilde{x} \otimes \tilde{x}$. Unlike the quadratic case, in which pairs of spectral components are coupled, the cubic term mixes triplets of spectral components.

Assuming again that the linear response contains only the frequencies of the external forces,

$$\tilde{x}(\omega) \sim a_1 \delta(\omega - \omega_1) + a_1^* \delta(\omega + \omega_1) + a_2 \delta(\omega - \omega_2) + a_2^* \delta(\omega + \omega_2), \quad (29)$$

we observe that the triple convolution produces spectral components of the form

$$\omega = \pm\omega_a \pm \omega_b \pm \omega_c, \quad (30)$$

where ω_a , ω_b , and ω_c can take the values ω_1 or ω_2 . Examples include

$$3\omega_1 = \omega_1 + \omega_1 + \omega_1, \quad (31)$$

$$2\omega_1 - \omega_2 = \omega_1 + \omega_1 + (-\omega_2), \quad (32)$$

$$2\omega_2 - \omega_1 = \omega_2 + \omega_2 + (-\omega_1). \quad (33)$$

These components correspond, respectively, to third-harmonic generation and to the characteristic sidebands of four-wave mixing. As in the quadratic case, successive nonlinear interactions lead to cascaded spectral generation. In a system with purely cubic nonlinearity, the resulting frequencies can be written as

$$\omega = n\omega_1 + m\omega_2, \quad |n| + |m| = 1, 3, 5, \dots \quad (34)$$

indicating that only odd-order combinations of the fundamental frequencies are generated.

Table 1 summarizes some representative nonlinear frequency-generation processes and their corresponding terminology in nonlinear optics.

Table 1: Examples of nonlinear frequency-generation processes discussed in this work and their optical analogs.

Process	Generated frequency	Optical analog
$\omega_1 + \omega_1$	$2\omega_1$	Second-harmonic generation (SHG)
$\omega_1 - \omega_1$	0	Optical rectification (OR)
$\omega_1 + \omega_2$	$\omega_1 + \omega_2$	Sum-frequency generation (SFG)
$\omega_1 - \omega_2$	$ \omega_1 - \omega_2 $	Difference-frequency generation (DFG)
$\omega_1 + \omega_1 + \omega_1$	$3\omega_1$	Third-harmonic generation (THG)
$2\omega_1 - \omega_2$	$2\omega_1 - \omega_2$	Four-wave mixing (FWM)
$2\omega_2 - \omega_1$	$2\omega_2 - \omega_1$	Four-wave mixing (FWM)

3. Examples

In this section, we present examples of the dynamics of the nonlinear forced mass-spring oscillator subjected to one or two external driving forces. The cases discussed here were solved numerically using the Mass-Spring System simulator (Fig. 2) available on the SimuFísica platform [8–10]. The numerical algorithm employed in the simulator is based on the fourth-order Runge-Kutta method [14].

3.1. Second-harmonic generation

In nonlinear optics, second-harmonic generation is a second-order process in which the medium absorbs two photons with frequency ω_1 and emits a photon with frequency $2\omega_1$. This process has several technological applications, including laser frequency conversion, nonlinear microscopy, and the generation of radiation in spectral regions that are difficult to access directly.

In Fig. 3, we present a numerical example dominated by the quadratic nonlinearity, that is, with $k_3 = 0$. We consider $k_2 = k/100 = 0.9 \text{ N/m}^2$ and only one external force, $F_1(t)$ ($F_2 = 0$), oscillating with amplitude $F_1 = 100 \text{ N}$ and angular frequency $\omega_1 = 4 \text{ rad/s}$. The remaining parameters are $\gamma = 1 \text{ kg}\cdot\text{s}^{-1}$, $m = 10 \text{ kg}$, and $k = 90 \text{ N/m}$, such that the natural frequency of the system is $\omega_0 = 3 \text{ rad/s}$. The system therefore operates

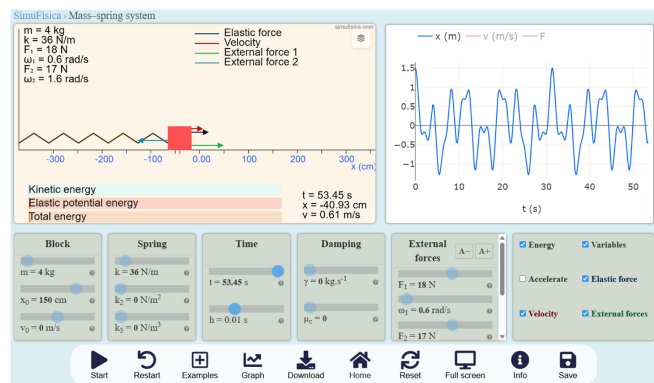


Figure 2: Graphical interface of the Mass-Spring System simulator (version 1.13) used to investigate the cases discussed in this work. Access link: <https://simufisica.com/en/spring-mass/>.

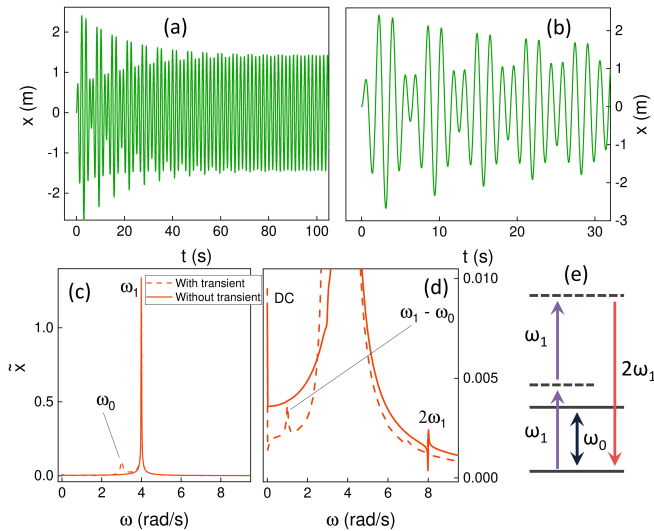


Figure 3: Dynamics of the nonlinear mass-spring system and its spectral components. (a) Temporal evolution of the mass position x . (b) Zoom of the temporal evolution in the interval $t \in [0, 32]$ s. (c) Fourier transform of x with (dashed curve) and without (solid curve) the transient regime. (d) Zoom of the Fourier transform. (e) Schematic representation of the optical/quantum interpretation, in the photon picture, of second-harmonic generation. Parameters used: $m = 10$ kg, $k = 90$ N/m, $\gamma = 1$ kg·s⁻¹, $k_2 = 0.9$ N/m², $k_3 = 0$, $F_1 = 100$ N, $\omega_1 = 4$ rad/s, and $F_2 = 0$. Access link for this case: <https://simufisica.com/95KQ3B>.

outside the resonance condition. For these parameters, the potential $U(x)$ has a local maximum, marking the edge of the locally stable region discussed in Sec. 2.1, at $x^* = -k/k_2 = -100$ m. The oscillation amplitudes obtained in this simulation remain well within this range. A small, positive k_3 term could alternatively be retained to render the potential globally bounded while keeping the quadratic nonlinearity dominant in this regime.

As initial conditions, we assume that the mass is initially at rest at the equilibrium position at $t = 0$. Figure 3(a) shows the temporal evolution of the position $x(t)$ of the mass. An initial transient regime associated with the damped natural response of the oscillator is observed. After approximately $t \approx 90$ s, the dynamics becomes dominated by the forced periodic response.

Figure 3(b) presents a zoom of the temporal evolution in the interval $t \in [0, 32]$ s. In this region, a clear amplitude modulation associated with the beat phenomenon [15] can be observed, with beat frequency given by $\omega_1 - \omega_0 = 1$ rad/s. This behavior arises from the interference between the natural response of the oscillator, near ω_0 , and the external driving frequency ω_1 .

Figure 3(c) shows the Fourier transform of $x(t)$ considering the entire temporal evolution, $t \in [0, 304.19]$ s (60838 points, dashed curve), and also only the stationary regime after removing the initial transient ($t \in [94.97, 304.19]$ s, 41844 points, solid curve). In the dashed curve, two main spectral components are

observed: the external driving frequency and the damped natural frequency of the oscillator, close to ω_0 . When the transient regime is removed, the response becomes dominated by the component associated with the external excitation.

The time step used in the numerical integration was $h = 0.005$ s, and the Fourier transform shown here, and throughout this section, was computed via the FFT algorithm implemented in the FFTW library [16], using a rectangular window, amplitude-type window correction, and the electrical-engineering sign convention; the amplitude spectrum is the one displayed in all cases. We note that the exact length of the time window affects, through the spectral leakage associated with the rectangular window, the apparent width of the spectral peaks, but not their frequency position, consistent with representing the response by Dirac delta functions being an idealization, as already noted in Sec. 2.2.

A zoom of the Fourier spectrum is shown in Fig. 3(d), revealing additional spectral components of the system. At $\omega = 8$ rad/s, a clear second-harmonic component appears at $2\omega_1$. A spectral component at $|\omega_1 - \omega_0|$ is also present, disappearing after the transient regime is removed. This component should not be identified with the beat phenomenon of Fig. 3(b): the linear superposition of two tones at ω_0 and ω_1 has a Fourier transform with peaks only at ω_0 and ω_1 themselves, the beat being a time-domain envelope-modulation effect. The spectral component at $|\omega_1 - \omega_0|$ observed here is instead generated by the quadratic nonlinearity, which mixes the transient (near ω_0) and forced (ω_1) contributions to the response into a genuine difference-frequency component; it disappears once the transient is removed because one of the two mixed frequencies is then absent, not because a beat envelope vanishes. This is confirmed numerically: repeating the same simulation with $k_2 = 0$ (linear system), while keeping the same transient and the same time window, both $2\omega_1$ and $|\omega_1 - \omega_0|$ disappear from the spectrum, leaving only ω_0 and ω_1 , as expected, since a linear superposition of these two tones has no spectral component at their sum or difference. A companion component at $\omega_0 + \omega_1$ is also observed, likewise vanishing when $k_2 = 0$, consistent with the same quadratic mixing mechanism. In addition, a DC component appears at $\omega = 0$, as predicted in Sec. 2.2. In the context of nonlinear optics, this component is analogous to optical rectification, in the sense that the nonlinear response generates a static (DC) contribution to the polarization; whether this actually produces a measurable static field depends on the sample geometry, boundary conditions, and charge redistribution, aspects not addressed by the present model. The interpretation of second-harmonic generation in terms of photons is illustrated in Fig. 3(e).

When the mass-spring oscillator enters resonance with the external force, the signatures of second-harmonic generation and the analog of optical rectification become more pronounced. In Fig. 4, we present the

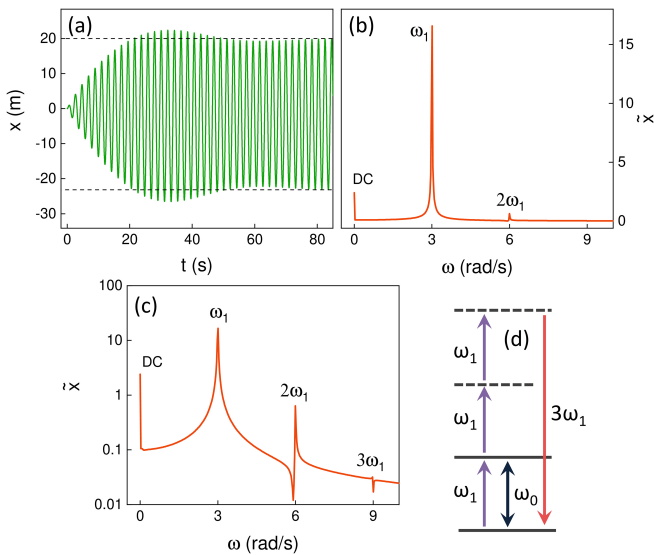


Figure 4: Dynamics of the nonlinear mass-spring system and its spectral components under resonance conditions. (a) Temporal evolution of the mass position x . (b) Fourier transform of x . (c) Fourier transform on a semilogarithmic scale. (d) Schematic representation of the optical/quantum interpretation, in the photon picture, of third-harmonic generation. Parameters used: the same as in Fig. 3, except that $\omega_1 = \omega_0 = 3$ rad/s. The Fourier transform was computed over $t \in [80.61, 305.995]$ s (45077 points), after removing the initial transient. Access link for this case: <https://simufisica.com/8PI2fF>.

results obtained under the same conditions as in Fig. 3, except that now $\omega_1 = \omega_0 = 3$ rad/s.

The DC component of the oscillation can be directly observed in Fig. 4(a) through the asymmetry between maxima and minima after the transient regime (see dashed line), and also in Fig. 4(b), where the relative amplitude of the Fourier component at $\omega = 0$ becomes significantly larger. Figure 4(c), displayed on a semilogarithmic scale, also reveals the presence of the third harmonic.

Figure 4(d) presents a photon-based interpretation of the weak spectral component at $3\omega_1$. Since $k_3 = 0$ in this example, this component is not a direct cubic response. As noted above, it instead arises from successive nonlinear interactions mediated by the quadratic term.

3.2. Four-wave mixing

In nonlinear optics, four-wave mixing is a third-order process in which different spectral components interact through the cubic susceptibility $\chi^{(3)}$ of the medium. In its simplest form, in the photon picture, two photons at frequency ω_1 are annihilated and one photon at frequency ω_2 and one at the sideband frequency $\omega_{\text{FWM}} = 2\omega_1 - \omega_2$ are created, so that energy conservation reads $2\omega_1 = \omega_2 + \omega_{\text{FWM}}$ (and analogously for the $2\omega_2 - \omega_1$ sideband). This photon-picture description is a quantum-mechanical interpretation and is not itself a consequence of the classical mechanical dynamics considered here.

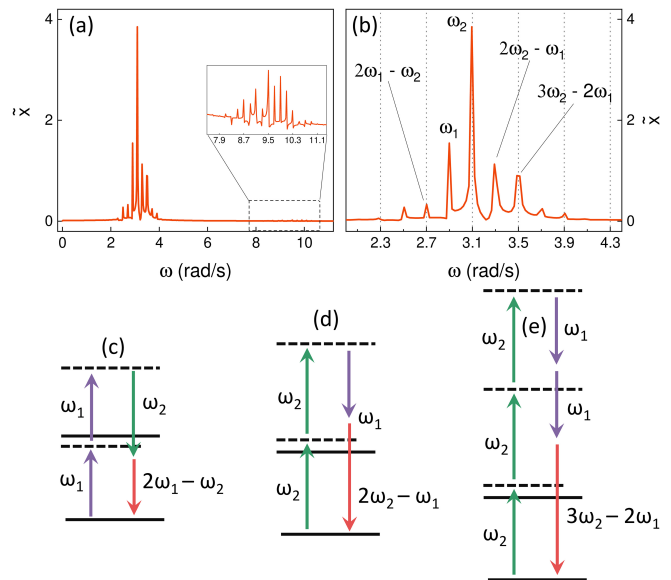


Figure 5: Dynamics of the nonlinear mass-spring system with $k_3 \neq 0$ and two external forces. (a) Fourier transform of x . Inset: zoom in the region $\omega \in [7.7, 11.1]$ rad/s. (b) Zoom of (a) in the region $\omega \in [2.1, 4.4]$ rad/s. (c), (d), and (e) represent the optical/quantum interpretation, in the photon picture, of some of the oscillation frequencies present in (b). Parameters used: the same as in Fig. 3, except that $\omega_1 = 2.9$ rad/s, $\omega_2 = 3.1$ rad/s, $F_1 = F_2 = 50$ N, $k_2 = 0$, and $k_3 = 0.9$ N/m³. The Fourier transform was computed over $t \in [50.88, 306.665]$ s (51157 points), after removing the initial transient. Access link for this case: <https://simufisica.com/9DmLd4>.

This mechanism plays an important role in phenomena such as parametric amplification, frequency conversion, and optical frequency-comb generation.

In Fig. 5, we present the simulation results for the case in which $k_2 = 0$ and $k_3 = k/100 = 0.9$ N/m³. Two external forces are considered, with amplitudes $F_1 = F_2 = 50$ N and angular frequencies $\omega_1 = 2.9$ rad/s and $\omega_2 = 3.1$ rad/s. The remaining parameters and initial conditions are the same as those used in Figs. 3 and 4, such that the natural frequency of the system remains $\omega_0 = 3$ rad/s.

Figure 5(a) shows the Fourier transform of the position $x(t)$, while Fig. 5(b) presents a zoom in the region $\omega \in [2.2, 4.4]$ rad/s. Several frequencies associated with third-order nonlinear processes can be observed. In particular, the frequencies

$$2\omega_1 - \omega_2 = 2.7 \text{ rad/s} \quad (35)$$

and

$$2\omega_2 - \omega_1 = 3.3 \text{ rad/s}, \quad (36)$$

corresponding to the characteristic sidebands of four-wave mixing, are clearly visible and schematically represented in Figs. 5(c) and 5(d), respectively.

Once formed, the modes at 2.7 and 3.3 rad/s may participate in further nonlinear interactions, giving rise

to additional frequencies through cascade processes. One example is the component

$$3\omega_2 - 2\omega_1 = 3.5 \text{ rad/s}, \quad (37)$$

illustrated in Fig. 5(e).

Figure 5(b) also reveals a comb-like spectral structure composed of approximately equally spaced modes separated by the difference between the driving frequencies,

$$\omega_2 - \omega_1 = 0.2 \text{ rad/s}. \quad (38)$$

This spectral pattern is analogous, in its equal spacing, to the mode structure produced by cascaded four-wave-mixing processes in optical frequency combs [4]. We stress that this analogy is limited to this frequency spacing: a genuine optical frequency comb further requires stable phase coherence between the modes, sustained by an underlying locking mechanism, neither of which is addressed by the present classical model.

In addition to the main FWM sidebands around the driving frequencies, a weaker spectral structure emerges near the third-harmonic region, extending approximately from 8.1 to 10.9 rad/s, as shown in the inset of Fig. 5(a). These peaks are associated with third-order nonlinear processes involving components such as

$$3\omega_1, \quad (39)$$

$$2\omega_1 + \omega_2, \quad (40)$$

$$\omega_1 + 2\omega_2, \quad (41)$$

and

$$3\omega_2, \quad (42)$$

in addition to higher-order cascaded interactions. While the terms $3\omega_1$ and $3\omega_2$ correspond to third-harmonic generation, the intermediate mixed frequencies $2\omega_1 + \omega_2$ and $\omega_1 + 2\omega_2$ are more generally classified as third-order wave-mixing processes associated with the cubic nonlinearity. We reserve the specific terms third-harmonic generation and four-wave mixing for the particular processes identified as such throughout this work.

4. Conclusions

In this work, we investigated the dynamics of a nonlinear mass-spring oscillator driven by two external periodic forces and explored its connection with nonlinear optical wave-mixing processes. By introducing quadratic and cubic nonlinear terms in the restoring force, we showed that the oscillator generates new spectral components through frequency-mixing mechanisms analogous, at the level of spectral-frequency combinations and nonlinear-response order, to second- and third-order nonlinear optical phenomena.

Using Fourier-domain arguments, we demonstrated how the quadratic nonlinearity produces second-order processes such as second-harmonic generation, optical rectification, sum-frequency generation, and difference-frequency generation. Likewise, the cubic nonlinearity was shown to generate third-order processes, including third-harmonic generation and four-wave mixing sidebands. The repeated action of these nonlinear mixing mechanisms leads naturally to cascaded spectral generation and to the emergence of frequency-comb-like structures.

The numerical simulations obtained with the interactive SimuFísica mass-spring simulator reproduced the spectral signatures predicted by the theoretical model. In particular, the simulations revealed the appearance of harmonic generation, sidebands associated with four-wave mixing, beat frequencies, DC components, and cascaded nonlinear spectral structures. The results also illustrated the importance of resonance conditions and transient dynamics in the observed spectra. Beyond confirming these expected consequences of the nonlinear model, the main contribution of this work lies in establishing, within a single controllable computational framework, a broader and more systematic correspondence between a simple mechanical analogy and second- and third-order nonlinear-optical processes than that offered by previous mechanical analogies restricted to parametric three-wave interactions [6], thereby providing a validated and extensible tool for exploring and teaching these phenomena.

Beyond the specific nonlinear processes investigated here, the model provides an accessible classical framework for discussing fundamental concepts of nonlinear optics using a mechanical analogy familiar to undergraduate students. The direct visualization of the oscillator motion and its Fourier spectrum offers an intuitive way to connect nonlinear differential equations, spectral analysis, and wave-mixing phenomena. In this sense, the simulator may serve as a useful pedagogical tool in courses on oscillations and introductory nonlinear optics. This pedagogical potential is presented here as a possible application of the model and its computational implementation. An assessment of its actual educational effectiveness, for instance through classroom implementation and evaluation of student learning, remains an open direction for future work.

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Data Availability

The datasets generated during this work are openly available in the Zenodo repository at <https://doi.org/10.5281/zenodo.20487214>.

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